Non-transitive Dice

Kiah Han Mao 99S21

Intuition *'the only real valuable thing(Einstein)'* may let us down at times as things are not what they seem. Consider the following group of dice [1] :

Dice A with faces labelled: 5, 5, 5, 5, 1, 1.
Dice B with faces labelled: 4, 4, 4, 3, 3, 3.
Dice C with faces labelled: 6, 6, 2, 2, 2, 2.

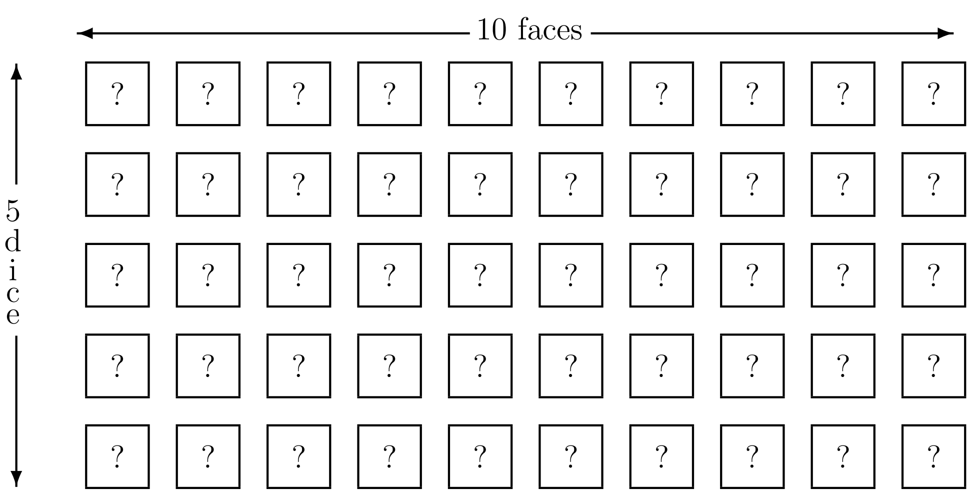
With some calculations, we find that of probability of dice A winning dice B is $\frac{2}{3}$. Moreover, the chances for B winning C, and C winning A is $\frac{2}{3}$ too. This means that A is likely to win B, B likely to win C and peculiarly, C likely to win A.

Contrast this with the usual transitive relation 'greater than'. If we say that $a > b$ and $b > c$, then it follows naturally that $a > c$. Shown above by a counter-example, we know that not all relationships are transitive. Hence, we call this group of dice non-transitive. This group of dice is also known as Efron's dice.

However, is this group of dice unique? Can we have non-transitive groups with any number of dice which have any number of faces?

Here, we come out with a simple algorithm which can generate groups of $m(m \geq 3)$ dice having $n(n \geq 2)$ faces.

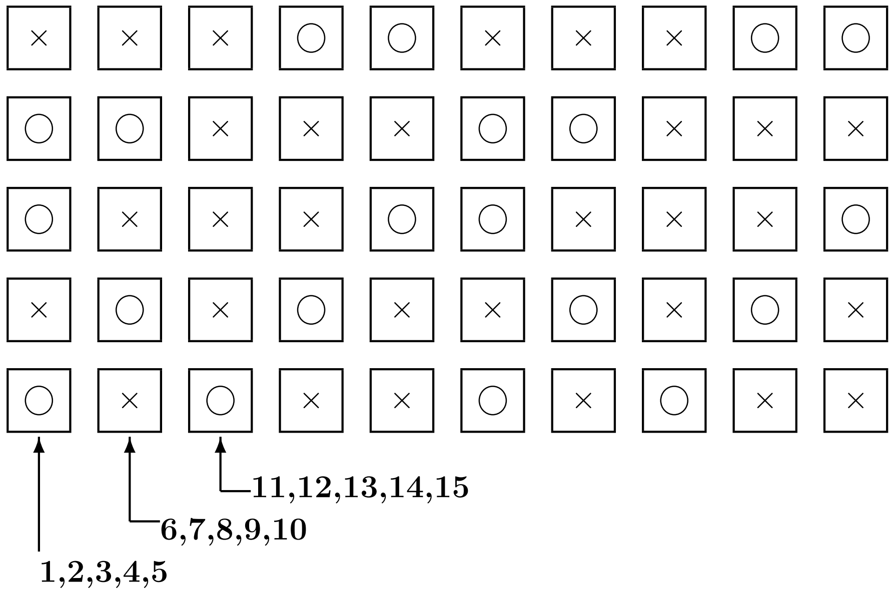
Let us consider the case of 5 dice with 10 faces. We represent the faces with squares and arrange the 5 dice into rows as shown below.



So we are going to fill up the individual boxes with numbers so that the following result holds: "probability of the 1st die winning the 2nd = probability of 2nd winning the 3rd = ... = probability of the 5th winning the 1st"

Before we put in the numbers, let us fill the squares with circles and crosses with two simple rules, *that no column contain all crosses or all circles* and *that all the rows contain the same number of crosses.*

[1] David Green, (1993), Puzzling Probability Problems, Mathematics Teacher, pg 18-19



Now, we are going to fill in the numbers into the squares, abiding to the following rules: *the numbers 1 to 5 goes into the first column, 6 to 10 to the second, 11 to 15 to the third, and so on,* and *where the cell contains a ◯, the number is smaller than the number directly below it, and where the cell contains a ×, the number is larger than the number directly below it (for the last row, the 'number directly below it' is in the first row).*

And one possible arrangement is:

5	8	15	16	21	30	33	40	41	46
1	6	14	20	25	26	31	39	45	50
2	10	13	19	22	27	35	38	44	47
4	7	12	17	24	29	32	37	42	49
3	9	11	18	23	28	34	36	43	48

And we have our group of non-transitive dice! the first dice having the numbers 5, 8, 15, 16, 21... 46 and the second having 1, 6, 14, 20... 50 and so on.

The reader can verify for himself that the dice are really non-transitive.